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Quark flavour mixing and the exponential form of the Kobayashi–Maskawa matrix

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Abstract. The form of the mixing matrix of quarks is discussed. The exponential parameterisation of the Cabibbo–Kobayashi–Maskawa (CKM) matrix is identified as the most general form of the mixing matrix. Its properties are reviewed in the context of the unitarity requirement. A recurrence series representation of the exponential form of the mixing matrix that is easy to handle is obtained, allowing for a direct and simple method of calculation of the CKM matrix. The generation of the new parameterisations of the CKM matrix by the exponential form is demonstrated.

1 Introduction

One of the greatest advances in physics in the 20th century was certainly that of the formulation of the standard model (SM) [1–3], in which a joint description of the electromagnetic and weak interactions is implemented via the gauge theory based on the $\mathrm{SU}(2)_{L(\text{weak isospin})} \times \mathrm{U}(1)_{Y(\text{weak hypercharge})}$ group and its spontaneous breaking via the Higgs mechanism [1, 2, 4]. The SM has been extensively tested during the last decades, and so far experimental results do not contradict it. At present the Higgs boson remains the only one missing piece of the standard model to be confirmed experimentally.

The quark mass eigenstates differ from the weak eigenstates. These latter consist of a linear combination of eigenstates of the strong interaction. Indeed, the weak eigenstates of the quarks (q') are linear superpositions of the mass eigenstates (q) given by the unitary transformations:

$$\begin{pmatrix} u' \\ c' \\ t' \end{pmatrix}_{\mathrm{L,R}} = U_{\mathrm{L,R}} \begin{pmatrix} u \\ c \\ t \end{pmatrix}_{\mathrm{L,R}} ,$$
$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_{\mathrm{L,R}} = D_{\mathrm{L,R}} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_{\mathrm{L,R}} , \tag{1}$$

where the $U_{\mathrm{L,R}}$ and $D_{\mathrm{L,R}}$ are unitary matrices (unitary in order to preserve the form of the kinetic terms in the initial massless Dirac Lagrangian for the quarks), and the subscripts L and R stand for the left and right components. The matrices $U_{\mathrm{L,R}}$ and $D_{\mathrm{L,R}}$ diagonalise the mass matrices $M^{U,D}$ for the up and down quarks u and d:

$$U_{\mathrm{R}}^{-1} M^U U_{\mathrm{L}} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} ,$$
$$D_{\mathrm{R}}^{-1} M^D D_{\mathrm{L}} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} , \tag{2}$$

which constitutes the following mass terms for the up and the down quarks:

$$\begin{pmatrix} \bar{u}' & \bar{c}' & \bar{t}' \end{pmatrix}_{\mathrm{R}} M^U \begin{pmatrix} u' \\ c' \\ t' \end{pmatrix}_{\mathrm{L}} + \mathrm{h.c.} ,$$
$$\begin{pmatrix} \bar{d}' & \bar{s}' & \bar{b}' \end{pmatrix}_{\mathrm{R}} M^D \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_{\mathrm{L}} + \mathrm{h.c.} \tag{3}$$

In this case, the charged weak current for the three quark generations is written as follows:

$$I_\mu^{\mathrm{C}} = \begin{pmatrix} \bar{u}' & \bar{c}' & \bar{t}' \end{pmatrix}_{\mathrm{L}} \gamma_\mu \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_{\mathrm{L}}$$
$$= \begin{pmatrix} \bar{u} & \bar{c} & \bar{t} \end{pmatrix}_{\mathrm{L}} (U_{\mathrm{L}}^{+} D_{\mathrm{L}}) \gamma_\mu \begin{pmatrix} d \\ s \\ b \end{pmatrix}_{\mathrm{L}} . \tag{4}$$

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From the above formula, we pick out the mass mixing matrix V, as follows:

$$V \equiv U_{\mathrm{L}}^{+} D_{\mathrm{L}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} . \quad (5)$$

In this context it is clear that the mixing matrix V – a unitary 3×3 matrix that acts on the charge $-e/3$ physical mass states (d, s, b) and transforms them into new interaction eigenstates (d', s', b') – plays the central role in the theory.

2 Exponential and other mixing matrix parameterisations

The mixing matrix V can be written in several forms. The first one, proposed by Kobayashi and Maskawa [5], in explicit parameterisation was given by the Euler angles θ_i and the complex phase δ. The common form of this parameterisation with a different placement of the phase, given by the three mixing angles θ_{12}, θ_{23}, θ_{13} and the CP non-invariant phase δ, is written as follows [6–10]:

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\mathrm{e}^{-\mathrm{i}\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{23}c_{13} \end{pmatrix} , \quad (6)$$

with $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$, and the "generation" index $i, j = 1, 2, 3$. These parameterisations are exact to all orders. When one of the angles vanishes, so does the mixing between the related generations. In the limit $\theta_{23} = \theta_{13} = 0$ the third generation decouples from the other ones, and we end up with the well known case of Cabibbo mixing [11] with $\theta_{12} = \theta_{\mathrm{c}}$.

The proposals for the approximate form of the V matrix are essentially based on its expansion into a power series of the parameter $\lambda = \sin\theta_{\mathrm{c}} \approx 0.22$. The analysis of the experimental data related to the decay times of the various particles expressed in terms of the Cabibbo angle shows that the matrix V can be written with an accuracy of up to λ^2 in the following simple form:

$$V \cong \begin{pmatrix} 1-\lambda^2/2 & \lambda & 0 \\ -\lambda & 1-\lambda^2/2 & \lambda^2 \\ 0 & -\lambda^2 & 1 \end{pmatrix} . \quad (7)$$

The matrix V in (7) is unitary and sets the second order parameterisation. Among the higher order parameterisations of the mixing matrix, the one best known is the parameterisation proposed by Wolfenstein, which contains the free parameters ρ, η and A [12]:

$$V_{\mathrm{W}} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - \mathrm{i}\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-\mathrm{i}\eta) & -A\lambda^2 & 1 \end{pmatrix} . \quad (8)$$

It satisfies the unitarity requirement with an accuracy of $O(\lambda^3)$. Note that the complex entries of the matrix (8) can be written as exponents; then (8) is sometimes called the Altarelli–Franzini form.

Probably, the most general form of the mass mixing matrix is the exponential representation [13–15]:

$$\hat{V} = \exp\left(\hat{A}\right) , \quad (9)$$

where $\hat{A}$ is a 3×3 matrix. According to the Cayley–Hamilton theorem [16], the matrix V can be explicitly written as the sum of powers of the matrix $\hat{A}$, as follows:

$$\hat{V} = \mathrm{e}^{\hat{A}} = \alpha_0\hat{I} + \alpha_1\hat{A} + \alpha_2\hat{A}^2 , \quad (10)$$

where the coefficients of the expansion are determined from the algebraic system in which the eigenvalues of $\hat{A}$ are involved. In general, the entries of the matrix A are complex in order to account for the CP violation effects:

$$\hat{A} = \begin{pmatrix} 0 & a_1 & a_3 \\ -a_1^* & 0 & -a_2^* \\ -a_3^* & a_2 & 0 \end{pmatrix} . \quad (11)$$

In the simplest case, of real values of the entries, they can be set as follows:

$$a_i = \lambda^i,\ i = 1, 2, 3 , \quad \lambda = \sqrt{m_u/m_s}. \quad (12)$$

In order to account for the CP violation together with the flavour mixing, the matrix A can be written in the following simple anti-Hermitian form [17, 18], which ensures the unitarity of the matrix V:

$$\hat{A} = \begin{pmatrix} 0 & \lambda & \lambda^3\exp(\mathrm{i}\delta) \\ -\lambda & 0 & -\lambda^2 \\ -\lambda^3\exp(-\mathrm{i}\delta) & \lambda^2 & 0 \end{pmatrix} , \quad (13)$$

where the parameter δ accounts for the violation of the CP symmetry, and the parameter λ is responsible for the quark flavour mixing. Evidently, the free parameters in the parameterisation form (13) have been reduced, respectively, to that employed in the Cabibbo–Kobayashi–Maskawa matrix. Due to the small value of λ, the exponential representation (9) can be written in the form of a rapidly converging series:

$$\exp\left[\hat{A}\right] = \sum_{n=0}^{\infty} \frac{\hat{A}^n}{n!} . \quad (14)$$

The exponential form of the matrix V includes the commonly used Wolfenstein and Altarelli–Franzini matrices. Indeed, we can consider the following generalised form of the exponential matrix with additional parameters x and y:

$$\tilde{A} = \begin{pmatrix} 0 & \lambda & (x\lambda)^3\exp(\mathrm{i}\delta) \\ -\lambda & 0 & -(y\lambda)^2 \\ -(x\lambda)^3\exp(-\mathrm{i}\delta) & (y\lambda)^2 & 0 \end{pmatrix} , \quad (15)$$

which, upon expansion into a series to the fourth order of λ, yields

$$\tilde{V} \cong \begin{pmatrix} 1-\frac{1}{2}\lambda^2+\frac{\lambda^4}{24} & \lambda-\frac{\lambda^3}{6} & \lambda^3 A\,(\rho-\mathrm{i}\eta) \\ -\lambda+\frac{\lambda^3}{6} & 1-\frac{\lambda^2}{2}-\frac{\lambda^4}{2}\left(A^2-\frac{1}{12}\right) & \lambda^2 A\,(1-\alpha+\mathrm{i}\kappa) \\ \lambda^3 A\,(1-\rho-\mathrm{i}\eta) & -\lambda^2 A\,(1-\beta+\mathrm{i}\kappa) & 1-\frac{1}{2}A^2\lambda^4 \end{pmatrix}, \tag{16}$$

with

$$\rho = 1/2-\frac{x^3}{y^2}\cos\delta\,, \quad \eta=\frac{x^3}{y^2}\sin\delta\,, \quad \kappa=\frac{\lambda^2}{2}\eta\,,$$
$$A=-y^2\,,$$
$$\alpha=\frac{\lambda^2}{2}\left(\frac{1}{3}-\frac{x^3}{y^2}\cos\delta\right), \quad \beta=\frac{\lambda^2}{2}\left(\frac{1}{3}+\frac{x^3}{y^2}\cos\delta\right). \tag{17}$$

It is now obvious that keeping only the terms of third order in λ in (16) and taking into account the small value of $\lambda \approx 0.22$, we obtain the extended form of the Wolfenstein matrix [12, 19], which differs from the Altarelli–Franzini matrix just by the complex number representation $r\exp[\mathrm{i}\phi]=\rho+\mathrm{i}\eta$. However, different from [12], we write the parameters $\alpha,\beta,\rho,\eta,\kappa$ and μ in the matrix (16) as certain well defined functions of λ,x,y and δ, as given by the formulae (17). For $x=y=1$, i.e. from (13), exactly the form (8) of the Wolfenstein mixing matrix follows. The high order parameterisations, like for example in [20], are reproduced by (16) exactly up to third order, and the difference appears in the high order corrections. For example, there are corrections of fourth order in λ, ΔV_{cb}, ΔV_{cd} and ΔV_{us} in (16), and the coefficients in the terms ΔV_{cs} and ΔV_{ts} differ from the ones given in [20]. The imaginary part of ΔV_{ts} can be written $\mathrm{Im}\,(\Delta V_{ts})=-\mathrm{i}A\lambda^4\eta/2$, whereas ΔV_{ts} coincides with the corresponding term in [20].

3 Recurrence relations for the exponential CKM matrix

The mass mixing matrix V, written in the exponential form (9), can be calculated with theoretically arbitrary accuracy by accounting for high order terms in the power series expansion (14). In what follows, we shall demonstrate how this procedure can be completed. Consider the matrix of third order with complex entries $A=\{a_{ij}\}$ $(i,j=1,2,3)$ and the following invariants:

$$J_1=\mathrm{Tr}\,A=a_{11}+a_{22}+a_{33}\,, \tag{18}$$

$$J_2=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix}+\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33}\end{vmatrix}+\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{vmatrix} \tag{19}$$

and

$$J_3=\det A\,. \tag{20}$$

It has been shown [21] that, with the help of the Cayley–Hamilton theorem, the nth power of the matrix A can be expressed in terms of the Chebyshev polynomials of two variables and in terms of the first and the second powers of the matrix A as follows:

$$A^n = p_{n-1}(u,v)\,J_3^{\frac{n-2}{3}}A^2+\left(-vp_{n-2}(u,v)+p_{n-3}(u,v)\right) \times J_3^{\frac{n-1}{3}}A+p_{n-2}(u,v)\,J_3^{\frac{n}{3}}I\,. \tag{21}$$

Here the variables u and v are defined by the relations

$$u=J_1J_3^{-\frac{1}{3}}\,, \quad v=J_2J_3^{-\frac{2}{3}}\,, \tag{22}$$

I is the unit matrix, and the matrix invariants $J_{1,2,3}$ are defined by (18), (19), and (20). The Chebyshev polynomials of two variables $p_n(u,v)$ obey the following recurrence relation:

$$p_n(u,v)=up_{n-1}(u,v)-vp_{n-2}(u,v)+p_{n-3}(u,v)\,, \tag{23}$$

and this yields the first three terms:

$$p_0(u,v)=0, \quad p_1(u,v)=1, \quad p_2(u,v)=u\,. \tag{24}$$

In the case of the matrix A as given by (13) with $\delta=0$ (real case), the first and the third matrix invariants vanish,

$$J_3=0, \quad J_1=0, \quad J_2=\lambda^2+\lambda^4+\lambda^6\,. \tag{25}$$

Moreover, since $\det A=0$, the recurrence relation (21) reduces to the following simple form:

$$A^{n+1}=J_2^{\frac{n-1}{2}}U_{n-1}\left(\frac{J_1}{2\sqrt{J_2}}\right)A^2-J_2^{\frac{n}{2}}U_{n-2}\left(\frac{J_1}{2\sqrt{J_2}}\right)A\,, \tag{26}$$

where $U_n(x)$ are the Chebyshev polynomials of the second kind. Since $\mathrm{Tr}\,(A)=0$, the argument u (see (22)) of the Chebyshev polynomial equals zero, and the polynomial reduces to

$$U_n=\cos\left(\pi\frac{n}{2}\right), \tag{27}$$

with the following oscillating values: $U_n=1,0,-1,0,1,0,$ etc. Now, substituting the expressions (25), (26), (27), and (13) with $\delta=0$ into the series expansion (14), we obtain the following series for the matrix V:

$$\hat{V}=\hat{I}+\sum_{n=0}^{\infty}\frac{\left(\lambda^2+\lambda^4+\lambda^6\right)^{\frac{n}{2}}}{(n+1)!} \times\left[\cos\left(\frac{\pi n}{2}\right)\hat{A}\Big|_{\delta=0}+\frac{\sin\left(\frac{\pi n}{2}\right)\hat{A}^2\Big|_{\delta=0}}{\sqrt{(\lambda^2+\lambda^4+\lambda^6)}}\right], \tag{28}$$

where the matrix A is given by (13), and its second power explicitly can be written as follows:

$$\hat{A}^2=\begin{pmatrix} -\lambda^2-\lambda^6 & \lambda^5\exp(\mathrm{i}\delta) & -\lambda^3 \\ \lambda^5\exp(-\mathrm{i}\delta) & -\lambda^2-\lambda^4 & -\lambda^4\exp(\mathrm{i}\delta) \\ -\lambda^3 & -\lambda^4\exp(-\mathrm{i}\delta) & -\lambda^4-\lambda^6 \end{pmatrix}. \tag{29}$$

Thus, formula (28) essentially represents the exponential mass mixing matrix V, written in terms of the first and the second powers of the matrix $\hat{A}$ for $\delta = 0$.

In the case of complex values of the entries of $\hat{A}$, i.e. $\delta \neq 0$, the expression for the invariant J_2 coincides with that obtained for the real case. However, since $\det(\hat{A}) \neq 0$, J_3 takes a nonzero value,

$$J_3 = \det\left(\hat{A}\right) = -2\lambda^6 \sinh\left(\mathrm{i}\delta\right), \tag{30}$$

so that the recurrence relation in the form (21) has to be employed for our calculations. However, the relation $J_1 = 0$ makes the variable u vanish $-$ $u = 0$ (see the definition (22)), and hence, the mixing matrix V is now given by the following recurrence relation:

$$\hat{V} = \hat{I} + \hat{A} + \frac{\hat{A}^2}{2} + \sum_{n=3}^{\infty} \frac{1}{n!}\left(q_{n-1}\left(w\right) J_3^{\frac{n-2}{3}} \hat{A}^2 + \left(-v q_{n-2}\left(w\right) + q_{n-3}\left(w\right)\right) J_3^{\frac{n-1}{3}} \hat{A} + q_{n-2}\left(w\right) J_3^{\frac{n}{3}} \hat{I} \right), \tag{31}$$

where the matrix A is specified by (13) and its second order is given by (29), whereas the polynomials q_n are given by

$$q_n\left(w\right) = \frac{-w^{\frac{n+2}{2}} U_n\left(-\frac{1}{2w\sqrt{w}}\right) + w^{\frac{n+5}{2}} U_{n-1}\left(-\frac{1}{2w\sqrt{w}}\right) + w^{-n+1}}{2 + w^3}. \tag{32}$$

The polynomials in (32) involve Chebyshev polynomials of the second kind, $U_n(x)$, with variable the w related to the variable v (see the definition of (22)) by the following relation:

$$v = w - w^{-2}. \tag{33}$$

In the complex case, $\delta \neq 0$, the variable v can be written in terms of the parameters λ and δ, as it follows from (22), (30), and (25):

$$v = \left(\lambda^2 + \lambda^4 + \lambda^6\right)\left(-2\lambda^6 \sinh\left(\mathrm{i}\delta\right)\right)^{\frac{2}{3}}. \tag{34}$$

4 Conclusions

It can be seen from the exponential forms (13) and (15) that the matrix $\hat{V}$ can be represented as a sum of the following two terms, i.e., A_1 of the first order and A_2 of the second order in λ,

$$\hat{A} = \hat{A}_1 + \hat{A}_2, \quad A_1 = \begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & (x\lambda)^3 \exp\left(\mathrm{i}\delta\right) \\ 0 & 0 & -(y\lambda)^2 \\ -(x\lambda)^3 \exp\left(-\mathrm{i}\delta\right) & (y\lambda)^2 & 0 \end{pmatrix}. \tag{35}$$

Recalling that it is the exponential form (9) that gives physical sense to the matrix A, we can write $\hat{V}$ in the following form:

$$\hat{V} = \exp\left(\hat{A}\right) = \exp\left(\hat{A}_1 + \hat{A}_2\right) = \exp\left(\frac{\hat{A}_1}{2}\right)\exp\left(\hat{A}_2\right)\exp\left(\frac{\hat{A}_1}{2}\right) + \frac{1}{4!}\left[\hat{A}_1\left[\hat{A}_1\hat{A}_2\right]\right] + o\left(\lambda^6\right). \tag{36}$$

Taking into account the physical value of the parameter $\lambda \cong 0.22$, we find that the last commutator term in (36) is of the order $O(\lambda^6)$. The above relation (36) between the exponential forms yields in fact the new parameterisation with the mass mixing matrix being the following matrix product:

$$\hat{V}_{xy} = P_1 \cdot P_2 \cdot P_1, \quad P_1 = \exp\left(\frac{\hat{A}_1}{2}\right), \quad P_2 = \exp\left(\hat{A}_2\right), \tag{37}$$

which has the same entries as the matrix $\hat{V}$ in (9) up to $O(\lambda^6)$. The form of the matrix P_1 can easily be obtained from (37). Indeed,

$$P_1 = \begin{pmatrix} \cos\left(\lambda/2\right) & \sin\left(\lambda/2\right) & 0 \\ -\sin\left(\lambda/2\right) & \cos\left(\lambda/2\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{38}$$

and it can easily be verified that it is contained in the first order expansion of the Kobayashi–Cabibbo–Maskawa matrix (6). The matrix P_2, accounting for the second and higher order contributions, can be written as follows:

$$P_2 = \begin{pmatrix} \frac{Y^2 + \tilde{\lambda}^2 \cos\left(\lambda^2 \tilde{\Lambda}\right)}{\tilde{\Lambda}^2} & \mathrm{e}^{\mathrm{i}\delta}\tilde{\lambda} Y \frac{\left(1 - \cos\left(\lambda^2 \tilde{\Lambda}\right)\right)}{\tilde{\Lambda}^2} & \mathrm{e}^{\mathrm{i}\delta} \frac{\tilde{\lambda} \sin\left(\lambda^2 \tilde{\Lambda}\right)}{\tilde{\Lambda}} \\ \mathrm{e}^{-\mathrm{i}\delta}\tilde{\lambda} Y \frac{\left(1 - \cos\left(\lambda^2 \tilde{\Lambda}\right)\right)}{\tilde{\Lambda}^2} & \frac{\tilde{\lambda}^2 + Y^2 \cos\left(\lambda^2 \tilde{\Lambda}\right)}{\tilde{\Lambda}^2} & -Y \frac{\sin\left(\lambda^2 \tilde{\Lambda}\right)}{\tilde{\Lambda}} \\ -\mathrm{e}^{-\mathrm{i}\delta}\tilde{\lambda} \frac{\sin\left(\lambda^2 \tilde{\Lambda}\right)}{\tilde{\Lambda}} & Y \frac{\sin\left(\lambda^2 \tilde{\Lambda}\right)}{\tilde{\Lambda}} & \cos\left(\lambda^2 \tilde{\Lambda}\right) \end{pmatrix}, \tag{39}$$

where

$$\tilde{\Lambda} = \sqrt{Y^2 + \tilde{\lambda}^2}, \quad Y = y^2, \quad \tilde{\lambda} = x^3 \lambda. \tag{40}$$

The expressions (37) and (38) are exact, and the direct check of the unitarity of the matrix $\hat{V}_{xy}$ given by (37) and (38), (39) confirms that the new parameterisation $\hat{V}_{xy} = P_1 \cdot P_2 \cdot P_1$ is unitary.

The expansion of the V_{xy} matrix in a power series of λ to the order $O(\lambda^2)$ yields, for x=1, y=1 and $\delta = 0$, the Kobayashi–Cabibbo–Maskawa matrix V_{KM}, whereas the higher order expansions include the Altarelli–Franzini and the Wolfenstein parameterisations.

Moreover, the series form of the exponential mass matrix obtained, with each term eventually reduced to the first and second powers of the matrix A and the sum of its minors, allows one to find them easily in a numerical form with any chosen accuracy. The unitarity of the matrix $\hat{V}$ is preserved at every order of λ. The exponential forms (9) of the mass mixing matrix V obtained with account of the formulae (13) and (15), (28), (31), and (37), generalise the Wolfenstein and Altarelli–Franzini parameterisations of the matrix $\hat{V}$. The symmetric and generic form of the obtained parameterisation formulae naturally includes the hierarchical feature of the Wolfenstein and Altarelli–Franzini parameterisations. It can be seen that the calculations of the series, obtained for the exponential mass mixing matrix, can easily be performed also in the case of $\delta \neq 0$, thus accounting for the violation of the CP symmetry at any desired order of λ with the invariance preserved.

Formulae (31), (32) and also (37)–(40) define new unitary mixing matrix parameterisations. Eventually, we have to say that despite significant progress in experimental physics in the last decades, the experimental check of high order contributions to the mass mixing matrix still remains beyond present experimental capabilities.

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